

# A $G$ -family of quandles and handlebody-knots

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## Abstract

We introduce the notion of a  $G$ -family of quandles which is an algebraic system whose axioms are motivated by handlebody-knot theory, and use it to construct invariants for handlebody-knots. Our invariant can detect the chiralities of some handlebody-knots including unknown ones.

## 1 Introduction

A quandle [11, 15] is an algebraic system whose axioms are motivated by knot theory. Carter, Jelsovsky, Kamada, Langford, and Saito [1] defined the quandle homology theory and quandle cocycle invariants for links and surface-links. The quandle chain complex in [1] is a subcomplex of the rack chain complex in [4]. The quandle cocycle invariant extracts information from quandle colorings by a quandle cocycle, and are used to detect the chirality of links in [3, 18].

In this paper, we introduce the notion of a  $G$ -family of quandles which is an algebraic system whose axioms are motivated by handlebody-knot theory, and use it to construct invariants for handlebody-knots. A *handlebody-knot* is a handlebody embedded in the 3-sphere. A handlebody-knot can be represented by its trivalent spine, and the first author, in [6], gave a list of local moves connecting diagrams of spatial trivalent graphs which represent equivalent handlebody-knots. The axioms of a  $G$ -family of quandles are derived from the local moves.

A  $G$ -family of quandles gives us not only invariants for handlebody-knots but also a way to handle a number of quandles at once. We see that a  $G$ -family of quandles is indeed a family of quandles associated with a group  $G$ . Any quandle is contained in some  $G$ -family of quandles as we see in Proposition 2.3. We introduce a homology theory for  $G$ -families of quandles. A cocycle of a  $G$ -family of quandles gives a family of cocycles of quandles. Thus it is efficient to find cocycles of a  $G$ -family of quandles, and indeed Nosaka [17] gave some cocycles together with a method to construct a cocycle of a  $G$ -family of quandles induced by a  $G$ -invariant group cocycle.

A  $G$ -family of quandles induces a quandle which contains all quandles forming the  $G$ -family of quandles as subquandles. This quandle, which we call the associated quandle, has a suitable structure to define colorings of a diagram of a handlebody-knot. Putting weights on colorings with a cocycle of a  $G$ -family of quandles, we define a quandle cocycle invariant for handlebody-knots. In

[7], the first and second authors defined quandle colorings and quandle cocycle invariants for handlebody-links by introducing the notion of an  $A$ -flow for an abelian group  $A$ . Quandle cocycle invariants we define in this paper are non-abelian versions of the invariants. A usual knot can be regarded as a genus one handlebody-knot by taking its regular neighborhood, and some knot invariants have been modified and generalized to construct invariants for handlebody-knots. In [10], the third and fourth authors defined symmetric quandle colorings and symmetric quandle cocycle invariants for handlebody-links by generalizing symmetric quandle cocycle invariants of classical knots given in [12, 13].

A table of genus two handlebody-knots with up to 6 crossings is given in [8], and the handlebody-knots  $0_1, \dots, 6_{16}$  in the table were proved to be mutually distinct by using the fundamental groups of their complements, quandle cocycle invariants in [7] and some topological arguments in [9, 14]. Our quandle cocycle invariant can distinguish the handlebody-knots  $6_{14}$  and  $6_{15}$  whose complements have isomorphic fundamental groups, and detect the chiralities of the handlebody-knots  $5_2, 5_3, 6_5, 6_9, 6_{11}, 6_{12}, 6_{13}, 6_{14}, 6_{15}$ . In particular, the chiralities of  $5_3, 6_5, 6_{11}$  and  $6_{12}$  were not known.

This paper is organized as follows. In Section 2, we give the definition of a  $G$ -family of quandles together with some examples. In Section 3, we describe colorings with a  $G$ -family of quandles for handlebody-links. We define the homology for a  $G$ -family of quandles in Section 4 and define several invariants for handlebody-links including quandle cocycle invariants in Section 5. In Section 6, we calculate quandle cocycle invariants for handlebody-knots with up to 6 crossings and show the chirality for some of the handlebody-knots. In Section 7, we prove that our invariants can be regarded as a generalization of the invariants defined in [7].

## 2 A $G$ -family of quandles

A *quandle* [11, 15] is a non-empty set  $X$  with a binary operation  $*$  :  $X \times X \rightarrow X$  satisfying the following axioms.

- For any  $x \in X$ ,  $x * x = x$ .
- For any  $x \in X$ , the map  $S_x : X \rightarrow X$  defined by  $S_x(y) = y * x$  is a bijection.
- For any  $x, y, z \in X$ ,  $(x * y) * z = (x * z) * (y * z)$ .

A *rack* is a non-empty set  $X$  with a binary operation  $*$  :  $X \times X \rightarrow X$  satisfying the second and third axioms. When we specify the binary operation  $*$  of a quandle (resp. rack)  $X$ , we denote the quandle (resp. rack) by the pair  $(X, *)$ . An *Alexander quandle*  $(M, *)$  is a  $\Lambda$ -module  $M$  with the binary operation defined by  $x * y = tx + (1 - t)y$ , where  $\Lambda := \mathbb{Z}[t, t^{-1}]$ . A *conjugation quandle*  $(G, *)$  is a group  $G$  with the binary operation defined by  $x * y = y^{-1}xy$ .

Let  $G$  be a group with identity element  $e$ . A  $G$ -family of quandles is a non-empty set  $X$  with a family of binary operations  $*^g : X \times X \rightarrow X$  ( $g \in G$ ) satisfying the following axioms.

- For any  $x \in X$  and any  $g \in G$ ,  $x *^g x = x$ .
- For any  $x, y \in X$  and any  $g, h \in G$ ,

$$x *^{gh} y = (x *^g y) *^h y \text{ and } x *^e y = x.$$

- For any  $x, y, z \in X$  and any  $g, h \in G$ ,

$$(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z).$$

When we specify the family of binary operations  $*^g : X \times X \rightarrow X$  ( $g \in G$ ) of a  $G$ -family of quandles, we denote the  $G$ -family of quandles by the pair  $(X, \{ *^g \}_{g \in G})$ .

**Proposition 2.1.** *Let  $G$  be a group. Let  $(X, \{ *^g \}_{g \in G})$  be a  $G$ -family of quandles.*

- (1) *For each  $g \in G$ , the pair  $(X, *^g)$  is a quandle.*
- (2) *We define a binary operation  $*$  :  $(X \times G) \times (X \times G) \rightarrow X \times G$  by*

$$(x, g) * (y, h) = (x *^h y, h^{-1}gh).$$

*Then  $(X \times G, *)$  is a quandle.*

We call the quandle  $(X \times G, *)$  in Proposition 2.1 the *associated quandle* of  $X$ . We note that the involution  $f : X \times G \rightarrow X \times G$  defined by  $f((x, g)) = (x, g^{-1})$  is a good involution of the associated quandle  $X \times G$ , where we refer the reader to [12] for the definition of a good involution of a quandle. Before proving this proposition, we introduce a notion of a  $Q$ -family of quandles. Let  $(Q, \triangleleft)$  be a quandle. A  $Q$ -family of quandles is a non-empty set  $X$  with a family of binary operations  $*^a : X \times X \rightarrow X$  ( $a \in Q$ ) satisfying the following axioms.

- For any  $x \in X$  and any  $a \in Q$ ,  $x *^a x = x$ .
- For any  $x \in X$  and any  $a \in Q$ , the map  $S_{x,a} : X \rightarrow X$  defined by  $S_{x,a}(y) = y *^a x$  is a bijection.
- For any  $x, y, z \in X$  and any  $a, b \in Q$ ,  $(x *^a y) *^b z = (x *^b z) *^{a \triangleleft b} (y *^b z)$ .

Let  $Q$  be a rack. A  $Q$ -family of racks is a non-empty set  $X$  with a family of binary operations  $*^a : X \times X \rightarrow X$  ( $a \in Q$ ) satisfying the second and third axioms.

**Lemma 2.2.** *Let  $(Q, \triangleleft)$  be a quandle (resp. rack). Let  $(X, \{\ast^a\}_{a \in Q})$  be a  $Q$ -family of quandles (resp. racks). We define a binary operation  $\ast : (X \times Q) \times (X \times Q) \rightarrow X \times Q$  by*

$$(x, a) \ast (y, b) = (x \ast^b y, a \triangleleft b).$$

*Then  $(X \times Q, \ast)$  is a quandle (resp. rack).*

*Proof.* The first axiom of a quandle follows from the equalities

$$(x, a) \ast (x, a) = (x \ast^a x, a \triangleleft a) = (x, a).$$

For any  $(x, a), (y, b) \in X \times Q$ , there is a unique  $(z, c) \in X \times Q$  such that  $x = z \ast^b y$  and  $a = c \triangleleft b$ . By the equalities  $(x, a) = (z \ast^b y, c \triangleleft b) = (z, c) \ast (y, b)$ , we have the second axiom of a quandle. The third axiom of a quandle follows from

$$\begin{aligned} ((x, a) \ast (y, b)) \ast (z, c) &= ((x \ast^b y) \ast^c z, (a \triangleleft b) \triangleleft c) \\ &= ((x \ast^c z) \ast^{b \triangleleft c} (y \ast^c z), (a \triangleleft c) \triangleleft (b \triangleleft c)) \\ &= ((x, a) \ast (z, c)) \ast ((y, b) \ast (z, c)). \end{aligned}$$

□

Conversely, we can prove the following. Let  $\triangleleft$  be a binary operation on a non-empty set  $Q$ . Let  $\ast^a$  be a binary operation on a non-empty set  $X$  for  $a \in Q$ . We define a binary operation  $\ast : (X \times Q) \times (X \times Q) \rightarrow X \times Q$  by

$$(x, a) \ast (y, b) = (x \ast^b y, a \triangleleft b).$$

If  $(X \times Q, \ast)$  is a quandle (resp. rack), then  $(Q, \triangleleft)$  is a quandle (resp. rack) and  $(X, \{\ast^a\}_{a \in Q})$  is a  $Q$ -family of quandles (resp. racks).

*Proof of Proposition 2.1.* (1) The first and third axioms of a quandle are easily checked. The second axiom of a quandle follows from the equalities

$$(x \ast^g y) \ast^{g^{-1}} y = (x \ast^{g^{-1}} y) \ast^g y = x.$$

Then  $(X, \ast^g)$  is a quandle.

(2) Let  $(G, \triangleleft)$  be the conjugation quandle. By Lemma 2.2,  $(X \times G, \ast)$  is a quandle.

□

The following proposition gives us many examples for a  $G$ -family of quandles.

**Proposition 2.3.** *(1) Let  $(X, \ast)$  be a quandle. Let  $S_x : X \rightarrow X$  be the bijection defined by  $S_x(y) = y \ast x$ . Let  $m$  be a positive integer such that  $S_x^m = \text{id}_X$  for any  $x \in X$  if such an integer exists. We define the binary operation  $\ast^i : X \times X \rightarrow X$  by  $x \ast^i y = S_y^i(x)$ . Then  $X$  is a  $\mathbb{Z}$ -family of quandles and a  $\mathbb{Z}_m$ -family of quandles, where  $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ .*

- (2) Let  $R$  be a ring, and  $G$  a group with identity element  $e$ . Let  $X$  be a right  $R[G]$ -module, where  $R[G]$  is the group ring of  $G$  over  $R$ . We define the binary operation  $*^g : X \times X \rightarrow X$  by  $x *^g y = xg + y(e - g)$ . Then  $X$  is a  $G$ -family of quandles.

*Proof.* (1) We verify the axioms of a  $G$ -family of quandles.

$$\begin{aligned} x *^0 y &= S_y^0(x) = \text{id}_X(x) = x, \\ x *^i x &= S_x^i(x) = x, \\ (x *^i y) *^j y &= S_y^j(S_y^i(x)) = S_y^{i+j}(x) = x *^{i+j} y. \end{aligned}$$

For the last axiom of a  $G$ -family of quandles, we can prove

$$(x *^j z) *^i (y *^j z) = (x *^i y) *^j z$$

by induction.

- (2) We verify the axioms of a  $G$ -family of quandles.

$$\begin{aligned} x *^e y &= xe + y(e - e) = x, \\ x *^g x &= xg + x(e - g) = x, \\ (x *^g y) *^h y &= (xg + y - yg)h + y - yh = x *^{gh} y, \end{aligned}$$

$$\begin{aligned} &(x *^h z) *^{h^{-1}gh} (y *^h z) \\ &= (xh + z - zh)h^{-1}gh + (yh + z - zh) - (yh + z - zh)h^{-1}gh \\ &= (xg + y - yg)h + z - zh \\ &= (x *^g y) *^h z. \end{aligned}$$

□

### 3 Handlebody-links and $X$ -colorings

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere  $S^3$ . Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of  $S^3$  which sends one to the other. A *spatial graph* is a finite graph embedded in  $S^3$ . Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of  $S^3$  which sends one to the other. When a handlebody-link  $H$  is a regular neighborhood of a spatial graph  $K$ , we say that  $K$  *represents*  $H$ , or  $H$  *is represented by*  $K$ . In this paper, a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link.

An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1, where the replacement is applied in a 3-ball embedded in  $S^3$ . Then we have the following theorem.

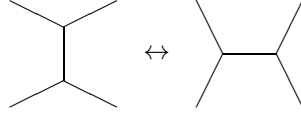


Figure 1:

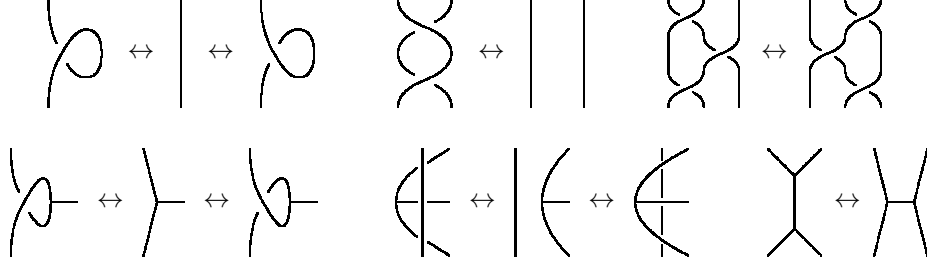


Figure 2:

**Theorem 3.1** ([6]). *For spatial trivalent graphs  $K_1$  and  $K_2$ , the following are equivalent.*

- $K_1$  and  $K_2$  represent an equivalent handlebody-link.
- $K_1$  and  $K_2$  are related by a finite sequence of IH-moves.
- Diagrams of  $K_1$  and  $K_2$  are related by a finite sequence of the moves depicted in Figure 2.

Let  $D$  be a diagram of a handlebody-link  $H$ . We set an orientation for each edge in  $D$ . Then  $D$  is a diagram of an oriented spatial trivalent graph  $K$ . We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by  $\pi/2$  on the diagram. We denote by  $\mathcal{A}(D)$  the set of arcs of  $D$ , where an arc is a piece of a curve each of whose endpoints is an undercrossing or a vertex. For an arc  $\alpha$  incident to a vertex  $\omega$ , we define  $\epsilon(\alpha; \omega) \in \{1, -1\}$  by

$$\epsilon(\alpha; \omega) = \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \omega, \\ -1 & \text{otherwise.} \end{cases}$$

Let  $X$  be a  $G$ -family of quandles, and  $Q$  the associated quandle of  $X$ . Let  $p_X$  (resp.  $p_G$ ) be the projection from  $Q$  to  $X$  (resp.  $G$ ). An  $X$ -coloring of  $D$  is a map  $C : \mathcal{A}(D) \rightarrow Q$  satisfying the following conditions at each crossing  $\chi$  and each vertex  $\omega$  of  $D$  (see Figure 3).

- Let  $\chi_1, \chi_2$  and  $\chi_3$  be respectively the under-arcs and the over-arc at a crossing  $\chi$  such that the normal orientation of  $\chi_3$  points from  $\chi_1$  to  $\chi_2$ . Then

$$C(\chi_2) = C(\chi_1) * C(\chi_3).$$

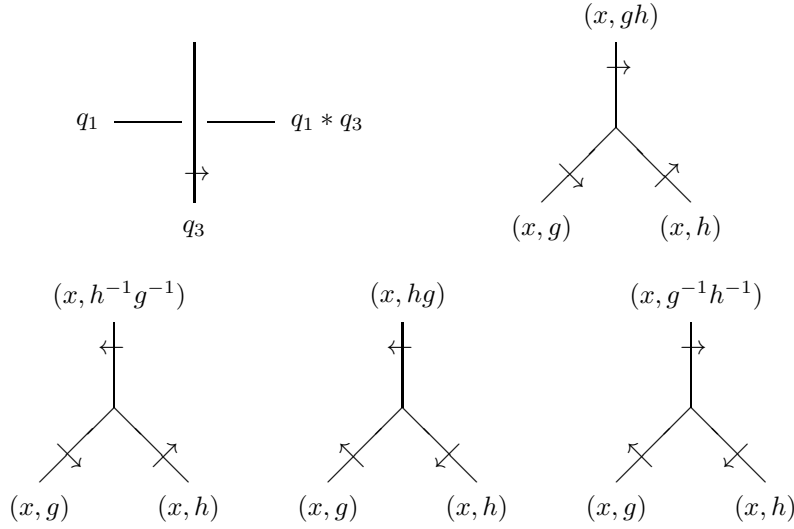


Figure 3:

- Let  $\omega_1, \omega_2, \omega_3$  be the arcs incident to a vertex  $\omega$  arranged clockwise around  $\omega$ . Then

$$(p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),$$

$$(p_G \circ C)(\omega_1)^{\epsilon(\omega_1; \omega)} (p_G \circ C)(\omega_2)^{\epsilon(\omega_2; \omega)} (p_G \circ C)(\omega_3)^{\epsilon(\omega_3; \omega)} = e.$$

We denote by  $\text{Col}_X(D)$  the set of  $X$ -colorings of  $D$ . We call  $C(\alpha)$  the *color* of  $\alpha$ . For two diagrams  $D$  and  $E$  which locally differ, we denote by  $\mathcal{A}(D, E)$  the set of arcs that  $D$  and  $E$  share.

**Lemma 3.2.** *Let  $X$  be a  $G$ -family of quandles. Let  $D$  be a diagram of an oriented spatial trivalent graph. Let  $E$  be a diagram obtained by applying one of the R1–R6 moves to the diagram  $D$  once, where we choose orientations for  $E$  which agree with those for  $D$  on  $\mathcal{A}(D, E)$ . For  $C \in \text{Col}_X(D)$ , there is a unique  $X$ -coloring  $C_{D,E} \in \text{Col}_X(E)$  such that  $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ .*

*Proof.* The color of an edge in  $\mathcal{A}(E) - \mathcal{A}(D, E)$  is uniquely determined by the colors of edges in  $\mathcal{A}(D, E)$ , since we have

$$a *^g a = a$$

for the R1, R4 moves, and

$$(a *^g b) *^{g^{-1}} b = a *^e b = a$$

for the R2 move, and

$$(a *^g b) *^h c = (a *^h c) *^{h^{-1}gh} (b *^h c)$$

for the R3 move, and

$$((b *^g a) *^h a) *^{(gh)^{-1}} a = a *^e b = b$$

for the R5 move, and only the coloring condition for the R6-move.  $\square$

Let  $X$  be a  $G$ -family of quandles. An  $X$ -set is a non-empty set  $Y$  with a family of maps  $*^g : Y \times X \rightarrow Y$  satisfying the following axioms, where we note that we use the same symbol  $*^g$  as the binary operation of the  $G$ -family of quandles.

- For any  $y \in Y$ ,  $x \in X$ , and any  $g, h \in G$ ,

$$y *^{gh} x = (y *^g x) *^h x \text{ and } y *^e x = y.$$

- For any  $y \in Y$ ,  $x_1, x_2 \in X$ , and any  $g, h \in G$ ,

$$(y *^g x_1) *^h x_2 = (y *^h x_2) *^{h^{-1}gh} (x_1 *^h x_2).$$

Any  $G$ -family of quandles  $(X, \{ *^g \}_{g \in G})$  itself is an  $X$ -set with its binary operations. Any singleton set  $\{y\}$  is also an  $X$ -set with the maps  $*^g$  defined by  $y *^g x = y$  for  $x \in X$  and  $g \in G$ , which is a trivial  $X$ -set.

Let  $D$  be a diagram of an oriented spatial trivalent graph. We denote by  $\mathcal{R}(D)$  the set of complementary regions of  $D$ . Let  $X$  be a  $G$ -family of quandles, and  $Y$  an  $X$ -set. Let  $Q$  be the associated quandle of  $X$ . An  $X_Y$ -coloring of  $D$  is a map  $C : \mathcal{A}(D) \cup \mathcal{R}(D) \rightarrow Q \cup Y$  satisfying the following conditions.

- $C(\mathcal{A}(D)) \subset Q$ ,  $C(\mathcal{R}(D)) \subset Y$ .
- The restriction  $C|_{\mathcal{A}(D)}$  of  $C$  on  $\mathcal{A}(D)$  is an  $X$ -coloring of  $D$ .
- For any arc  $\alpha \in \mathcal{A}(D)$ , we have

$$C(\alpha_1) * C(\alpha) = C(\alpha_2),$$

where  $\alpha_1, \alpha_2$  are the regions facing the arc  $\alpha$  so that the normal orientation of  $\alpha$  points from  $\alpha_1$  to  $\alpha_2$  (see Figure 4).

We denote by  $\text{Col}_X(D)_Y$  the set of  $X_Y$ -colorings of  $D$ .

For two diagrams  $D$  and  $E$  which locally differ, we denote by  $\mathcal{R}(D, E)$  the set of regions that  $D$  and  $E$  share. Since colors of regions are uniquely determined by those of arcs and one region, Lemma 3.2 implies the following lemma.

**Lemma 3.3.** *Let  $X$  be a  $G$ -family of quandles,  $Y$  an  $X$ -set. Let  $D$  be a diagram of an oriented spatial trivalent graph. Let  $E$  be a diagram obtained by applying one of the R1–R6 moves to the diagram  $D$  once, where we choose orientations for  $E$  which agree with those for  $D$  on  $\mathcal{A}(D, E)$ . For  $C \in \text{Col}_X(D)_Y$ , there is a unique  $X_Y$ -coloring  $C_{D,E} \in \text{Col}_X(E)_Y$  such that  $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$  and  $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$ .*



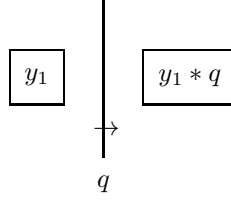


Figure 4:

## 4 A homology

Let  $X$  be a  $G$ -family of quandles, and  $Y$  an  $X$ -set. Let  $(Q, *)$  be the associated quandle of  $X$ . Let  $B_n(X)_Y$  be the free abelian group generated by the elements of  $Y \times Q^n$  if  $n \geq 0$ , and let  $B_n(X)_Y = 0$  otherwise. We put

$$((y, q_1, \dots, q_i) * q, q_{i+1}, \dots, q_n) := (y * q, q_1 * q, \dots, q_i * q, q_{i+1}, \dots, q_n)$$

for  $y \in Y$  and  $q, q_1, \dots, q_n \in Q$ . We define a boundary homomorphism  $\partial_n : B_n(X)_Y \rightarrow B_{n-1}(X)_Y$  by

$$\begin{aligned} \partial_n(y, q_1, \dots, q_n) &= \sum_{i=1}^n (-1)^i (y, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n) \\ &\quad - \sum_{i=1}^n (-1)^i ((y, q_1, \dots, q_{i-1}) * q_i, q_{i+1}, \dots, q_n) \end{aligned}$$

for  $n > 0$ , and  $\partial_n = 0$  otherwise. Then  $B_*(X)_Y = (B_n(X)_Y, \partial_n)$  is a chain complex (see [1, 2, 4, 5]).

Let  $D_n(X)_Y$  be the subgroup of  $B_n(X)_Y$  generated by the elements of

$$\bigcup_{i=1}^{n-1} \left\{ (y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n) \mid \begin{array}{l} y \in Y, x \in X, g, h \in G \\ q_1, \dots, q_n \in Q \end{array} \right\}$$

and

$$\bigcup_{i=1}^n \left\{ \begin{array}{l} (y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n) \\ -(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ -((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n) \end{array} \mid \begin{array}{l} y \in Y, x \in X, \\ g, h \in G, \\ q_1, \dots, q_n \in Q \end{array} \right\}.$$

We remark that

$$(y, q_1, \dots, q_{i-1}, (x, e), q_{i+1}, \dots, q_n)$$

and

$$\begin{aligned} &(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\ &+ ((y, q_1, \dots, q_{i-1}) * (x, g), (x, g^{-1}), q_{i+1}, \dots, q_n) \end{aligned}$$

belong to  $D_n(X)_Y$ .

**Lemma 4.1.** For  $n \in \mathbb{Z}$ , we have  $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$ . Thus  $D_*(X)_Y = (D_n(X)_Y, \partial_n)$  is a subcomplex of  $B_*(X)_Y$ .

*Proof.* It is sufficient to show the equalities

$$\begin{aligned} & \partial_n(y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n) = 0, \\ & \partial_n(y, q_1, \dots, q_{i-1}, (x, gh), q_{i+2}, \dots, q_n) \\ &= \partial_n(y, q_1, \dots, q_{i-1}, (x, g), q_{i+2}, \dots, q_n) \\ &+ \partial_n((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+2}, \dots, q_n) \end{aligned}$$

in  $B_{n-1}(X)_Y/D_{n-1}(X)_Y$ . We verify the first equality in the quotient group.

$$\begin{aligned} & \partial_n(y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_n) \\ &= (-1)^i(y, q_1, \dots, q_{i-1}, (x, h), q_{i+2}, \dots, q_n) \\ &+ (-1)^{i+1}(y, q_1, \dots, q_{i-1}, (x, g), q_{i+2}, \dots, q_n) \\ &- (-1)^i((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+2}, \dots, q_n) \\ &- (-1)^{i+1}((y, q_1, \dots, q_{i-1}, (x, g)) * (x, h), q_{i+2}, \dots, q_n) \\ &= (-1)^i(y, q_1, \dots, q_{i-1}, (x, h), q_{i+2}, \dots, q_n) \\ &+ (-1)^{i+1}(y, q_1, \dots, q_{i-1}, (x, gh), q_{i+2}, \dots, q_n) \\ &- (-1)^{i+1}((y, q_1, \dots, q_{i-1}) * (x, h), (x, h^{-1}gh), q_{i+2}, \dots, q_n) \\ &= 0, \end{aligned}$$

where the first equality follows from

$$((y, q_1, \dots, q_{i-1}, (x, g), (x, h), q_{i+2}, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_n) = 0.$$

We verify the second equality in the quotient group.

$$\begin{aligned}
& \partial_n(y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_n) \\
&= \sum_{j < i} (-1)^j (y, q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\
&+ \sum_{j < i} (-1)^j ((y, q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n) \\
&+ (-1)^i (y, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n) \\
&+ \sum_{j > i} (-1)^j (y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_{j-1}, q_{j+1}, \dots, q_n) \\
&+ \sum_{j > i} (-1)^j ((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_{j-1}, q_{j+1}, \dots, q_n) \\
&- \sum_{j < i} (-1)^j ((y, q_1, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\
&- \sum_{j < i} (-1)^j (((y, q_1, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n) \\
&- (-1)^i ((y, q_1, \dots, q_{i-1}) * (x, gh), q_{i+1}, \dots, q_n) \\
&- \sum_{j > i} (-1)^j ((y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_n) \\
&= \partial_n(y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_n) \\
&+ \partial_n((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_n),
\end{aligned}$$

where the last equality follows from

$$\begin{aligned}
& ((y, q_1, \dots, q_{i-1}) * (x, gh), q_{i+1}, \dots, q_n) \\
&= (((y, q_1, \dots, q_{i-1}) * (x, g)) * (x, h), q_{i+1}, \dots, q_n)
\end{aligned}$$

and

$$\begin{aligned}
& ((y, q_1, \dots, q_{i-1}, (x, gh), q_{i+1}, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_n) \\
&= ((y, q_1, \dots, q_{i-1}, (x, g), q_{i+1}, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_n) \\
&+ (((y, q_1, \dots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \dots, q_{j-1}) * q_j, q_{j+1}, \dots, q_n).
\end{aligned}$$

Then  $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$ .  $\square$

We put  $C_n(X)_Y = B_n(X)_Y / D_n(X)_Y$ . Then  $C_*(X)_Y = (C_n(X)_Y, \partial_n)$  is a chain complex. For an abelian group  $A$ , we define the cochain complex  $C^*(X; A)_Y = \text{Hom}(C_*(X)_Y, A)$ . We denote by  $H_n(X)_Y$  the  $n$ th homology group of  $C_*(X)_Y$ .

## 5 Cocycle invariants

Let  $X$  be a  $G$ -family of quandles, and  $Y$  an  $X$ -set. Let  $D$  be a diagram of an oriented spatial trivalent graph. For an  $X_Y$ -coloring  $C \in \text{Col}_X(D)_Y$ , we define

the weight  $w(\chi; C) \in C_2(X)_Y$  at a crossing  $\chi$  of  $D$  as follows. Let  $\chi_1, \chi_2$  and  $\chi_3$  be respectively the under-arcs and the over-arc at a crossing  $\chi$  such that the normal orientation of  $\chi_3$  points from  $\chi_1$  to  $\chi_2$ . Let  $R_\chi$  be the region facing  $\chi_1$  and  $\chi_3$  such that the normal orientations  $\chi_1$  and  $\chi_3$  point from  $R_\chi$  to the opposite regions with respect to  $\chi_1$  and  $\chi_3$ , respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where  $\epsilon(\chi) \in \{1, -1\}$  is the sign of a crossing  $\chi$ . We define a chain  $W(D; C) \in C_2(X)_Y$  by

$$W(D; C) = \sum_{\chi} w(\chi; C),$$

where  $\chi$  runs over all crossings of  $D$ .

**Lemma 5.1.** *The chain  $W(D; C)$  is a 2-cycle of  $C_*(X)_Y$ . Further, for cohomologous 2-cocycles  $\theta, \theta'$  of  $C^*(X; A)_Y$ , we have  $\theta(W(D; C)) = \theta'(W(D; C))$ .*

*Proof.* It is sufficient to show that  $W(D; C)$  is a 2-cycle of  $C_2(X)_Y$ . We denote by  $\mathcal{SA}(D)$  the set of curves obtained from  $D$  by removing (small neighborhoods of) crossings and vertices. We call a curve in  $\mathcal{SA}(D)$  a *semi-arc* of  $D$ . We note that a semi-arc is obtained by dividing an over-arc at all crossings. We denote by  $\mathcal{SA}(D; \chi)$  the set of semi-arcs incident to  $\chi$ , where  $\chi$  is a crossing or a vertex of  $D$ .

We define the orientation and the color of a semi-arc by those of the arc including the semi-arc. For a semi-arc  $\alpha$ , there is a unique region  $R_\alpha$  facing  $\alpha$  such that the orientation of  $\alpha$  points from the region  $R_\alpha$  to the opposite region with respect to  $\alpha$ . For a semi-arc  $\alpha$  incident to a crossing or a vertex  $\chi$ , we define

$$\epsilon(\alpha; \chi) := \begin{cases} 1 & \text{if the orientation of } \alpha \text{ points to } \chi, \\ -1 & \text{otherwise.} \end{cases}$$

Let  $\chi_1, \chi_2$  be the semi-arcs incident to a crossing  $\chi$  such that they originate from the under-arcs at  $\chi$  and that the normal orientation of the over-arc points from  $\chi_1$  to  $\chi_2$ . Let  $\chi_3, \chi_4$  be the semi-arcs incident to a crossing  $\chi$  such that they originate from the over-arc at  $\chi$  and that the normal orientation of the under-arcs points from  $\chi_3$  to  $\chi_4$  (see Figure 5). Then we have

$$\begin{aligned} \partial_2(w(\chi; C)) &= -\epsilon(\chi)(C(R_{\chi_1}), C(\chi_3)) + \epsilon(\chi)(C(R_{\chi_1}), C(\chi_1)) \\ &\quad + \epsilon(\chi)(C(R_{\chi_1}) * C(\chi_1), C(\chi_3)) \\ &\quad - \epsilon(\chi)(C(R_{\chi_1}) * C(\chi_3), C(\chi_1) * C(\chi_3)) \\ &= \sum_{\alpha \in \mathcal{SA}(D; \chi)} \epsilon(\alpha; \chi)(C(R_\alpha), C(\alpha)). \end{aligned}$$

Since  $\sum_{\alpha \in \mathcal{SA}(D; \omega)} \epsilon(\alpha; \omega)(C(R_\alpha), C(\alpha))$  is an element of  $D_1(X)_Y$  for a vertex

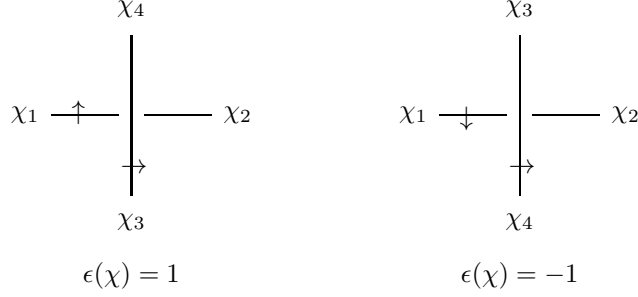


Figure 5:

$\omega$ , we have

$$\begin{aligned}
\partial_2 \left( \sum_{\chi} w(\chi; C) \right) &= \sum_{\chi} \sum_{\alpha \in \mathcal{SA}(D; \chi)} \epsilon(\alpha; \chi) (C(R_{\alpha}), C(\alpha)) \\
&= \sum_{\chi} \sum_{\alpha \in \mathcal{SA}(D; \chi)} \epsilon(\alpha; \chi) (C(R_{\alpha}), C(\alpha)) \\
&\quad + \sum_{\omega} \sum_{\alpha \in \mathcal{SA}(D; \omega)} \epsilon(\alpha; \omega) (C(R_{\alpha}), C(\alpha)) \\
&= \sum_{\alpha \in \mathcal{SA}(D)} ((C(R_{\alpha}), C(\alpha)) - (C(R_{\alpha}), C(\alpha))) \\
&= 0
\end{aligned}$$

in  $C_1(X)_Y$ , where  $\chi$  and  $\omega$  respectively run over all crossings and vertices of  $D$ .  $\square$

We recall that, for  $C \in \text{Col}_X(D)_Y$ , there is a unique  $X_Y$ -coloring  $C_{D,E} \in \text{Col}_X(E)$  such that  $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$  and  $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{A}(R,E)}$  by Lemma 3.3.

**Lemma 5.2.** *Let  $D$  be a diagram of an oriented spatial trivalent graph. Let  $E$  be a diagram obtained by applying one of the R1–R6 moves to the diagram  $D$  once, where we choose orientations for  $E$  which agree with those for  $D$  on  $\mathcal{A}(D, E)$ . For  $C \in \text{Col}_X(D)_Y$  and  $C_{D,E} \in \text{Col}_X(E)_Y$  such that  $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$  and  $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$ , we have  $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$ .*

*Proof.* We have the invariance under the R1, R4 and R5 moves, since the difference between  $[W(D; C)]$  and  $[W(E; C_{D,E})]$  is an element of  $D_2(X)_Y$ . The invariance under the R2 move follows from the signs of the crossings which appear in the move. We have the invariance under the R3 move, since the difference between  $[W(D; C)]$  and  $[W(E; C_{D,E})]$  is an image of  $\partial_3$ . We have the invariance under the R6 move, since no crossings appear in the move.  $\square$

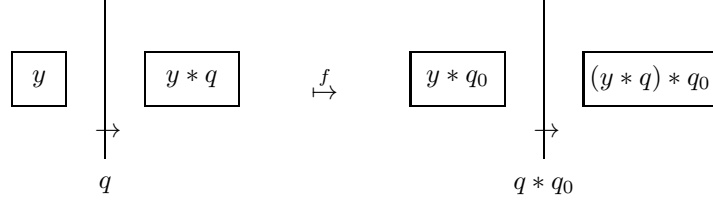


Figure 6:

We denote by  $G_H$  (resp.  $G_K$ ) the fundamental group of the exterior of a handlebody-link  $H$  (resp. a spatial graph  $K$ ). When  $H$  is represented by  $K$ , the groups  $G_H$  and  $G_K$  are identical. Let  $D$  be a diagram of an oriented spatial trivalent graph  $K$ . By the definition of an  $X_Y$ -coloring  $C$  of  $D$ , the map  $p_G \circ C|_{\mathcal{A}(D)}$  represents a homomorphism from  $G_K$  to  $G$ , which we denote by  $\rho_C \in \text{Hom}(G_K, G)$ . For  $\rho \in \text{Hom}(G_K, G)$ , we define

$$\text{Col}_X(D; \rho)_Y = \{C \in \text{Col}_X(D)_Y \mid \rho_C = \rho\}.$$

For a 2-cocycle  $\theta$  of  $C^*(X; A)_Y$ , we define

$$\begin{aligned} \mathcal{H}(D) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D)_Y\}, \\ \Phi_\theta(D) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D)_Y\}, \\ \mathcal{H}(D; \rho) &:= \{[W(D; C)] \in H_2(X)_Y \mid C \in \text{Col}_X(D; \rho)_Y\}, \\ \Phi_\theta(D; \rho) &:= \{\theta(W(D; C)) \in A \mid C \in \text{Col}_X(D; \rho)_Y\} \end{aligned}$$

as multisets.

**Lemma 5.3.** *Let  $D$  be a diagram of an oriented spatial trivalent graph  $K$ . For  $\rho, \rho' \in \text{Hom}(G_K, G)$  such that  $\rho$  and  $\rho'$  are conjugate, we have  $\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho')$  and  $\Phi_\theta(D; \rho) = \Phi_\theta(D; \rho')$ .*

*Proof.* Let  $g_0$  be an element of  $G$  such that  $\rho'(x) = g_0^{-1} \rho(x) g_0$  for any  $x \in G_K$ . Fix  $x_0 \in X$ . We set  $q_0 := (x_0, g_0)$ . Let  $f : \text{Col}_X(D; \rho)_Y \rightarrow \text{Col}_X(D; \rho')_Y$  be the bijection defined by  $f(C)(x) = C(x) * q_0$  (see Figure 6).

We prove  $[W(D; C)] = [W(D; f(C))]$  in  $H_2(X)_Y$  for  $C \in \text{Col}_X(D; \rho)_Y$ . We assume that spatial trivalent graphs are drawn in  $\mathbb{R}^2(\subset S^2)$ . Let  $D'$  be a diagram obtained from  $D$  by putting an oriented loop  $\gamma$  in the outermost region  $R_\infty$  so that the loop bounds a disk, where the loop is oriented counterclockwise (see Figure 7). Let  $C'$  be the  $X_Y$ -coloring of  $D'$  defined by  $C'(\gamma) = q_0$  and  $C' = C$  on  $\mathcal{A}(D, D') \cup \mathcal{R}(D, D')$ . Then we note that  $C'(R'_\infty) = C(R_\infty) * q_0$  for the region  $R'_\infty$  surrounded by the loop  $\gamma$  in  $D'$ . We deform the diagram  $D'$  by using R2, R3 and R5 moves so that the loop passes over all arcs of  $D$  exactly once. Then we denote by  $D''$  and  $C'' \in \text{Col}_X(D'')_Y$  the resulting diagram and the corresponding  $X_Y$ -coloring of  $D''$ , respectively. We obtain the  $X_Y$ -coloring  $f(C)$  from  $C''$  by removing the loop from  $D''$ , which also implies that  $f$  is well-defined.



Figure 7:

Since no crossings increase or decrease when we add or remove the loop  $\gamma$ , we have

$$[W(D; C)] = [W(D'; C')] = [W(D''; C'')] = [W(D; f(C))],$$

where the second equality follows from Lemma 5.2. Then we have  $\mathcal{H}(D; \rho) = \mathcal{H}(D; \rho')$  and  $\Phi_\theta(D; \rho) = \Phi_\theta(D; \rho')$ .  $\square$

We denote by  $\text{Conj}(G_K, G)$  the set of conjugacy classes of homomorphisms from  $G_K$  to  $G$ . By Lemma 5.3,  $\mathcal{H}(D; \rho)$  and  $\Phi_\theta(D; \rho)$  are well-defined for  $\rho \in \text{Conj}(G_K, G)$ .

**Lemma 5.4.** *Let  $D$  be a diagram of an oriented spatial trivalent graph  $K$ . Let  $E$  be a diagram obtained from  $D$  by reversing the orientation of an edge  $e$ . For  $\rho \in \text{Hom}(G_K, G)$ , we have  $\mathcal{H}(D) = \mathcal{H}(E)$ ,  $\Phi_\theta(D) = \Phi_\theta(E)$ ,  $\mathcal{H}(D; \rho) = \mathcal{H}(E; \rho)$  and  $\Phi_\theta(D; \rho) = \Phi_\theta(E; \rho)$ .*

*Proof.* It is sufficient to show that  $\mathcal{H}(D; \rho) = \mathcal{H}(E; \rho)$ . We define a bijection  $f : \text{Col}_X(D; \rho)_Y \rightarrow \text{Col}_X(E; \rho)_Y$  by  $f(C)(\alpha) = (p_X(C(\alpha)), p_G(C(\alpha))^{-1})$  if  $\alpha$  is an arc originates from the edge  $e$ ,  $f(C)(\alpha) = C(\alpha)$  otherwise. We remark that  $\rho_{f(C)} = \rho_C = \rho$ . The map  $f$  is well-defined, since  $z_1 * (x, g) = z_2$  is equivalent to  $z_2 * (x, g^{-1}) = z_1$ . Then we have  $w(\chi; C) = w(\chi; f(C))$  for every crossing  $\chi$ , since we have

$$\begin{aligned} (y, (x_1, g_1), (x_2, g_2)) &= -(y * (x_1, g_1), (x_1, g_1^{-1}), (x_2, g_2)) \\ &= -(y * (x_2, g_2), (x_1, g_1) * (x_2, g_2), (x_2, g_2^{-1})) \\ &= ((y * (x_1, g_1)) * (x_2, g_2), (x_1, g_1^{-1}) * (x_2, g_2), (x_2, g_2^{-1})) \end{aligned}$$

in  $C_2(X)_Y$  (see Figure 8). Then we have  $\mathcal{H}(D; \rho) = \mathcal{H}(E; \rho)$ .  $\square$

By Lemma 5.4,  $\mathcal{H}(D)$ ,  $\Phi_\theta(D)$ ,  $\mathcal{H}(D; \rho)$  and  $\Phi_\theta(D; \rho)$  are well-defined for a diagram  $D$  of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram  $D$  of a handlebody-link  $H$ , we define

$$\begin{aligned} \mathcal{H}^{\text{hom}}(D) &:= \{\mathcal{H}(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \\ \Phi_\theta^{\text{hom}}(D) &:= \{\Phi_\theta(D; \rho) \mid \rho \in \text{Hom}(G_H, G)\}, \\ \mathcal{H}^{\text{conj}}(D) &:= \{\mathcal{H}(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\}, \\ \Phi_\theta^{\text{conj}}(D) &:= \{\Phi_\theta(D; \rho) \mid \rho \in \text{Conj}(G_H, G)\} \end{aligned}$$

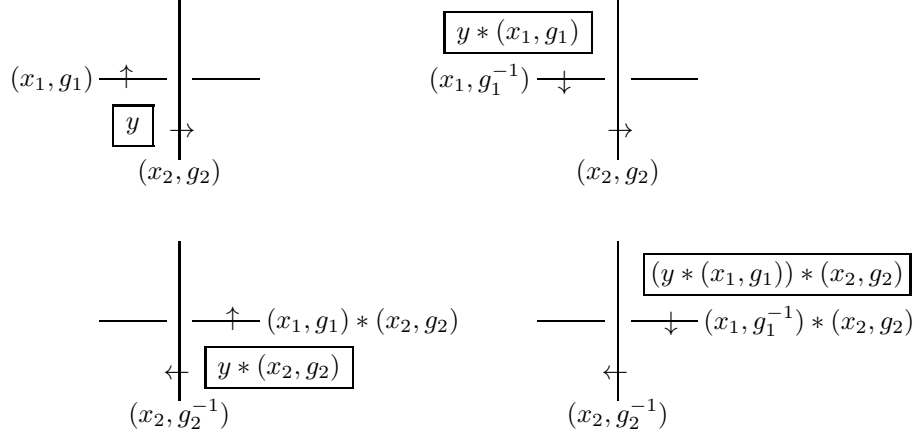


Figure 8:

as “multisets of multisets”. We remark that, for  $X_Y$ -colorings  $C$  and  $C_{D,E}$  in Lemma 5.2, we have  $\rho_C = \rho_{C_{D,E}}$ . Then, by Lemmas 5.1–5.4, we have the following theorem.

**Theorem 5.5.** *Let  $X$  be a  $G$ -family of quandles,  $Y$  an  $X$ -set. Let  $\theta$  be a 2-cocycle of  $C^*(X; A)_Y$ . Let  $H$  be a handlebody-link represented by a diagram  $D$ . Then the following are invariants of a handlebody-link  $H$ .*

$$\mathcal{H}(D), \quad \Phi_\theta(D), \quad \mathcal{H}^{\text{hom}}(D), \quad \Phi_\theta^{\text{hom}}(D), \quad \mathcal{H}^{\text{conj}}(D), \quad \Phi_\theta^{\text{conj}}(D).$$

We denote the invariants of  $H$  given in Theorem 5.5 by

$$\mathcal{H}(H), \quad \Phi_\theta(H), \quad \mathcal{H}^{\text{hom}}(H), \quad \Phi_\theta^{\text{hom}}(H), \quad \mathcal{H}^{\text{conj}}(H), \quad \Phi_\theta^{\text{conj}}(H),$$

respectively.

Let  $\{y\}$  be a trivial  $X$ -set. For the trivial 2-cocycle 0 of  $C^*(X; A)_{\{y\}}$ , we have

$$\begin{aligned} \Phi_0(H) &= \{0 \mid C \in \text{Col}_X(D)_{\{y\}}\}, \\ \Phi_0^{\text{hom}}(H) &= \{\{0 \mid C \in \text{Col}_X(D; \rho)_{\{y\}}\} \mid \rho \in \text{Hom}(G_H, G)\}, \\ \Phi_0^{\text{conj}}(H) &= \{\{0 \mid C \in \text{Col}_X(D; \rho)_{\{y\}}\} \mid \rho \in \text{Conj}(G_H, G)\}. \end{aligned}$$

Thus

$$\begin{aligned} \# \text{Col}_X(H) &:= \# \text{Col}_X(D)_{\{y\}}, \\ \# \text{Col}_X^{\text{hom}}(H) &:= \{\# \text{Col}_X(D; \rho)_{\{y\}} \mid \rho \in \text{Hom}(G_H, G)\}, \\ \# \text{Col}_X^{\text{conj}}(H) &:= \{\# \text{Col}_X(D; \rho)_{\{y\}} \mid \rho \in \text{Conj}(G_H, G)\} \end{aligned}$$



are invariants of a handlebody-link  $H$  represented by a diagram  $D$ , where  $\#S$  denotes the cardinality of a multiset  $S$ . We remark that these invariants do not depend on the choice of the singleton set  $\{y\}$ .

We denote by  $H^*$  the mirror image of a handlebody-link  $H$ . Then we have the following theorem.

**Theorem 5.6.** *For a handlebody-link  $H$ , we have*

$$\begin{aligned}\mathcal{H}(H^*) &= -\mathcal{H}(H), & \Phi_\theta(H^*) &= -\Phi_\theta(H), \\ \mathcal{H}^{\text{hom}}(H^*) &= -\mathcal{H}^{\text{hom}}(H), & \Phi_\theta^{\text{hom}}(H^*) &= -\Phi_\theta^{\text{hom}}(H), \\ \mathcal{H}^{\text{conj}}(H^*) &= -\mathcal{H}^{\text{conj}}(H), & \Phi_\theta^{\text{conj}}(H^*) &= -\Phi_\theta^{\text{conj}}(H),\end{aligned}$$

where  $-S = \{-a \mid a \in S\}$  for a multiset  $S$ .

*Proof.* Let  $D$  be a diagram of a handlebody-link  $H$ . We suppose that  $D$  is depicted in an  $xy$ -plane  $\mathbb{R}^2$ . Let  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the involution defined by  $\varphi(x, y) = (-x, y)$ . Let  $\tilde{\varphi} : S^3 \rightarrow S^3$  be the involution defined by  $\varphi(x, y, z) = (-x, y, z)$  and  $\varphi(\infty) = \infty$ , where we regard the 3-sphere  $S^3$  as  $\mathbb{R}^3 \cup \{\infty\}$ . Then  $\varphi(D)$  is a diagram of the handlebody-link  $H^* = \tilde{\varphi}(H)$ . For  $\rho \in \text{Hom}(G_H, G)$  and  $C \in \text{Col}_X(D; \rho)_Y$ , we have  $\tilde{\varphi}_*(\rho) \in \text{Hom}(G_{H^*}, G)$  and  $C \circ \varphi \in \text{Col}_X(\varphi(D); \tilde{\varphi}_*(\rho))_Y$ , where  $\tilde{\varphi}_*$  is the isomorphism induced by  $\tilde{\varphi}$ . For each crossing  $\chi$  of  $D$ ,  $\epsilon(\chi) = -\epsilon(\varphi(\chi))$ , and hence we have  $w(\varphi(\chi), C \circ \varphi) = -w(\chi, C)$ . Then  $[W(\varphi(D); C \circ \varphi)] = -[W(D; C)]$ , which implies the equalities in this theorem.  $\square$

## 6 Applications

In this section, we calculate cocycle invariants defined in the previous section for the handlebody-knots  $0_1, \dots, 6_{16}$  in the table given in [8], by using a 2-cocycle given by Nosaka [17]. This calculation enables us to distinguish some of handlebody-knots from their mirror images, and a pair of handlebody-knots whose complements have isomorphic fundamental groups.

Let  $G = SL(2; \mathbb{Z}_3)$  and  $X = (\mathbb{Z}_3)^2$ . Then  $X$  is a  $G$ -family of quandles with the proper binary operation as given in Proposition 2.3 (2). Let  $Y$  be the trivial  $X$ -set  $\{y\}$ . We define a map  $\theta : Y \times (X \times G)^2 \rightarrow \mathbb{Z}_3$  by

$$\theta(y, (x_1, g_1), (x_2, g_2)) := \lambda(g_1) \det(x_1 - x_2, x_2(1 - g_2^{-1})),$$

where the abelianization  $\lambda : G \rightarrow \mathbb{Z}_3$  is given by

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + d)(b - c)(1 - bc).$$

By [17], the map  $\theta$  is a 2-cocycle of  $C^*(X; \mathbb{Z}_3)_Y$ . Table 1 lists the invariant  $\Phi_\theta^{\text{conj}}(H)$  for the handlebody-knots  $0_1, \dots, 6_{16}$ . We represent the multiplicity of elements of a multiset by using subscripts. For example,  $\{\{0_2, 1_3\}_1, \{0_3\}_2\}$  represents the multiset  $\{\{0, 0, 1, 1, 1\}, \{0, 0, 0\}, \{0, 0, 0\}\}$ .

	$\Phi_\theta(H)$
0 <sub>1</sub>	$\{\{0_9\}_{76}\}$
4 <sub>1</sub>	$\{\{0_9\}_{83}, \{0_{27}\}_{22}, \{0_{81}\}_{3}\}$
5 <sub>1</sub>	$\{\{0_9\}_{76}\}$
5 <sub>2</sub>	$\{\{0_9\}_{95}, \{0_{27}\}_{6}, \{0_{81}\}_{1}, \{0_9, 1_{18}\}_{4}, \{0_{27}, 1_{54}\}_{2}\}$
5 <sub>3</sub>	$\{\{0_9\}_{102}, \{0_{27}\}_{4}, \{0_{27}, 2_{54}\}_{2}\}$
5 <sub>4</sub>	$\{\{0_9\}_{74}, \{0_{81}\}_{2}\}$
6 <sub>1</sub>	$\{\{0_9\}_{91}, \{0_{27}\}_{16}, \{0_{81}\}_{1}\}$
6 <sub>2</sub>	$\{\{0_9\}_{106}, \{0_{45}, 1_{18}, 2_{18}\}_{2}\}$
6 <sub>3</sub>	$\{\{0_9\}_{74}, \{0_{27}\}_{2}\}$
6 <sub>4</sub>	$\{\{0_9\}_{76}\}$
6 <sub>5</sub>	$\{\{0_9\}_{74}, \{0_9, 1_{18}\}_{2}\}$
6 <sub>6</sub>	$\{\{0_9\}_{72}, \{0_{27}\}_{4}\}$
6 <sub>7</sub>	$\{\{0_9\}_{85}, \{0_{27}\}_{16}, \{0_{81}\}_{3}, \{0_{45}, 1_{18}, 2_{18}\}_{4}\}$
6 <sub>8</sub>	$\{\{0_9\}_{76}\}$
6 <sub>9</sub>	$\{\{0_9\}_{91}, \{0_{27}\}_{6}, \{0_{81}\}_{1}, \{0_9, 1_{18}\}_{6}, \{0_{27}, 1_{54}\}_{2}, \{0_{27}, 2_{54}\}_{2}\}$
6 <sub>10</sub>	$\{\{0_9\}_{76}\}$
6 <sub>11</sub>	$\{\{0_9\}_{70}, \{0_9, 1_{18}\}_{6}\}$
6 <sub>12</sub>	$\{\{0_9\}_{97}, \{0_{81}\}_{1}, \{0_9, 1_{18}\}_{8}, \{0_9, 1_{36}, 2_{36}\}_{2}\}$
6 <sub>13</sub>	$\{\{0_9\}_{95}, \{0_{27}\}_{6}, \{0_{81}\}_{1}, \{0_9, 2_{18}\}_{4}, \{0_{27}, 2_{54}\}_{2}\}$
6 <sub>14</sub>	$\{\{0_9\}_{119}, \{0_{27}\}_{6}, \{0_{81}\}_{11}, \{0_9, 1_{18}\}_{12}, \{0_{27}, 1_{54}\}_{24}\}$
6 <sub>15</sub>	$\{\{0_9\}_{119}, \{0_{27}\}_{6}, \{0_{81}\}_{11}, \{0_9, 2_{18}\}_{12}, \{0_{27}, 1_{54}\}_{24}\}$
6 <sub>16</sub>	$\{\{0_9\}_{44}, \{0_{81}\}_{32}\}$

Table 1:

	chirality	M	II	LL	IKO	IIJO
0 <sub>1</sub>	○					
4 <sub>1</sub>	○					
5 <sub>1</sub>	×			✓		
5 <sub>2</sub>	×		✓	✓		✓
5 <sub>3</sub>	×					✓
5 <sub>4</sub>	×				✓	
6 <sub>1</sub>	×	✓				
6 <sub>2</sub>	?					
6 <sub>3</sub>	?					
6 <sub>4</sub>	×			✓		
6 <sub>5</sub>	×					✓
6 <sub>6</sub>	○					
6 <sub>7</sub>	○					
6 <sub>8</sub>	?					
6 <sub>9</sub>	×		✓			✓
6 <sub>10</sub>	?					
6 <sub>11</sub>	×					✓
6 <sub>12</sub>	×					✓
6 <sub>13</sub>	×		✓	✓		✓
6 <sub>14</sub>	×				✓	✓
6 <sub>15</sub>	×				✓	✓
6 <sub>16</sub>	○					

Table 2:

From Table 1, we see that our invariant can distinguish the handlebody-knots  $6_{14}$ ,  $6_{15}$ , whose complements have the isomorphic fundamental groups. Together with Theorem 5.6, we also see that handlebody-knots  $5_2$ ,  $5_3$ ,  $6_5$ ,  $6_9$ ,  $6_{11}$ ,  $6_{12}$ ,  $6_{13}$ ,  $6_{14}$ ,  $6_{15}$  are not equivalent to their mirror images. In particular, the chiralities of  $5_3$ ,  $6_5$ ,  $6_{11}$  and  $6_{12}$  were not known. Table 2 shows us known facts on the chirality of handlebody-knots in [8] so far. In the column of “chirality”, the symbols  $\bigcirc$  and  $\times$  mean that the handlebody-knot is amphichiral and chiral, respectively, and the symbol  $?$  means that it is not known whether the handlebody-knot is amphichiral or chiral. The symbols  $\checkmark$  in the right five columns mean that the handlebody-knots can be proved chiral by using the method introduced in the papers corresponding to the columns. Here, M, II, LL, IKO and IIJO denote the papers [16], [7], [14], [9] and this paper, respectively.

## 7 A generalization

In this section, we show that our invariant is a generalization of the invariant  $\Phi_\theta^I(H)$  defined by the first and second authors in [7]. We refer the reader to [7]

for the details of the invariant  $\Phi_\theta^I(H)$ . We recall the definition of the chain complex for the invariant  $\Phi_\theta^I(H)$ .

Let  $X$  be a  $\mathbb{Z}_m$ -family of quandles,  $Y$  an  $X$ -set. Let  $B_n^I(X)_Y$  be the free abelian group generated by the elements of  $Y \times X^n$  if  $n \geq 0$ , and let  $B_n^I(X)_Y = 0$  otherwise. We put

$$((y, x_1, \dots, x_i) *^j x, x_{i+1}, \dots, x_n) := (y *^j x, x_1 *^j x, \dots, x_i *^j x, x_{i+1}, \dots, x_n)$$

for  $y \in Y$ ,  $x, x_1, \dots, x_n \in X$  and  $j \in \mathbb{Z}_m$ . We define a boundary homomorphism  $\partial_n : B_n^I(X)_Y \rightarrow B_{n-1}^I(X)_Y$  by

$$\begin{aligned} \partial_n(y, x_1, \dots, x_n) &= \sum_{i=1}^n (-1)^i (y, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ &\quad - \sum_{i=1}^n (-1)^i ((y, x_1, \dots, x_{i-1}) *^1 x_i, x_{i+1}, \dots, x_n) \end{aligned}$$

for  $n > 0$ , and  $\partial_n = 0$  otherwise. Then  $B_*^I(X)_Y = (B_n^I(X)_Y, \partial_n)$  is a chain complex. Let  $D_n^I(X)_Y$  be the subgroup of  $B_n^I(X)_Y$  generated by the elements of

$$\bigcup_{i=1}^{n-1} \{(y, x_1, \dots, x_{i-1}, x, x, x_{i+2}, \dots, x_n) \mid y \in Y, x, x_1, \dots, x_n \in X\}$$

and

$$\bigcup_{i=1}^n \left\{ \sum_{j=0}^{m-1} ((y, x_1, \dots, x_{i-1}) *^j x_i, x_i, \dots, x_n) \mid y \in Y, x_1, \dots, x_n \in X \right\}.$$

Then  $D_*^I(X)_Y = (D_n^I(X)_Y, \partial_n)$  is a chain complex.

We put  $C_n^I(X)_Y = B_n^I(X)_Y / D_n^I(X)_Y$ . Then  $C_*^I(X)_Y = (C_n^I(X)_Y, \partial_n)$  is a chain complex. For an abelian group  $A$ , we define the cochain complex  $C_I^*(X; A)_Y = \text{Hom}(C_*^I(X)_Y, A)$ . We denote by  $H_n^I(X)_Y$  the  $n$ th homology group of  $C_*^I(X)_Y$ .

**Proposition 7.1.** *For  $n \in \mathbb{Z}$ , we have*

$$H_n^I(X)_Y \cong H_n(X)_Y.$$

*Proof.* The homomorphism  $f_n : C_n^I(X)_Y \rightarrow C_n(X)_Y$  defined by

$$f_n((y, x_1, \dots, x_n)) = (y, (x_1, 1), \dots, (x_n, 1))$$

is an isomorphism, since the homomorphism  $g_n : C_n(X)_Y \rightarrow C_n^I(X)_Y$  defined by

$$\begin{aligned} g_n(y, (x_1, s_1), \dots, (x_n, s_n)) \\ = \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \cdots \sum_{i_n=0}^{s_n-1} (\cdots ((y *^{i_1} x_1, x_1) *^{i_2} x_2, x_2) \cdots *^{i_n} x_n, x_n) \end{aligned}$$

is the inverse map of  $f_n$ . It is easy to see that  $f = \{f_n\}$  is a chain map from  $C_*^I(X)_Y$  to  $C_*(X)_Y$ . Therefore  $H_n^I(X)_Y \cong H_n(X)_Y$ .  $\square$

For a 2-cocycle  $\theta$  of  $C_1^*(X; A)_Y$ , the composition  $\theta \circ g_2$  is a 2-cocycle of  $C^*(X; A)_Y$ , and we have

$$\Phi_\theta^I(H) = \Phi_{\theta \circ g_2}^{\text{hom}}(H),$$

where  $g_2$  is the map defined in Proposition 7.1. Then our invariant is a generalization of the invariant introduced in [7].

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